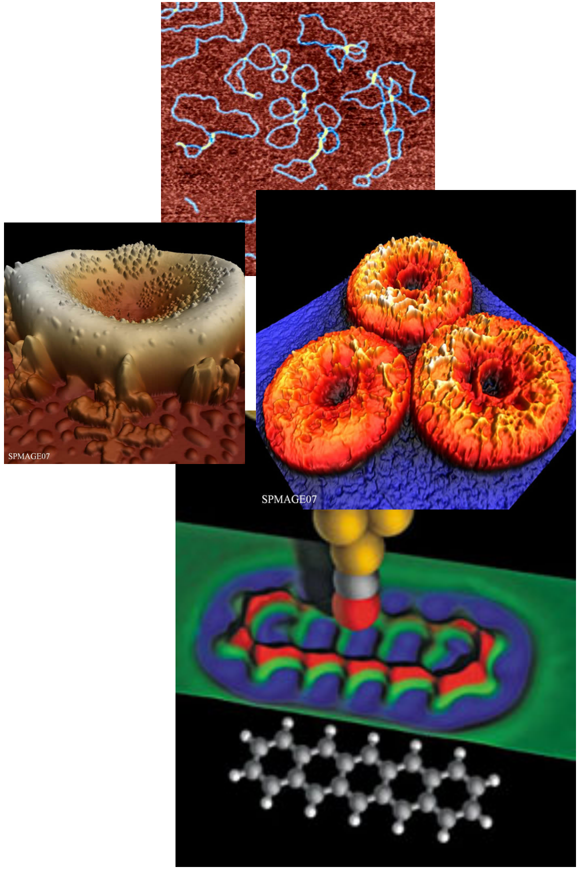
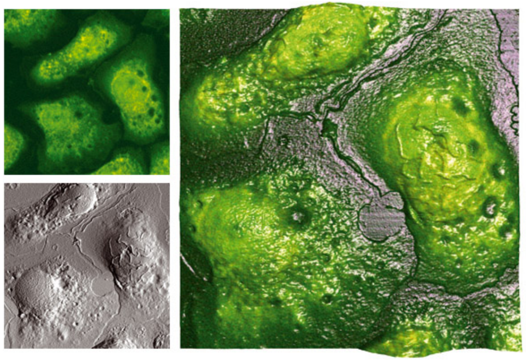
Lecture 13: Beam deflection

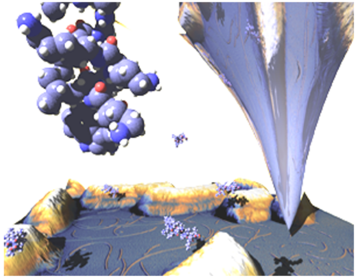
- Applications in nanotechnology
- Governing differential equations
- Solving beam deflection through integration
- Solving beam deflection through superposition
- Statically indeterminate beam deflection

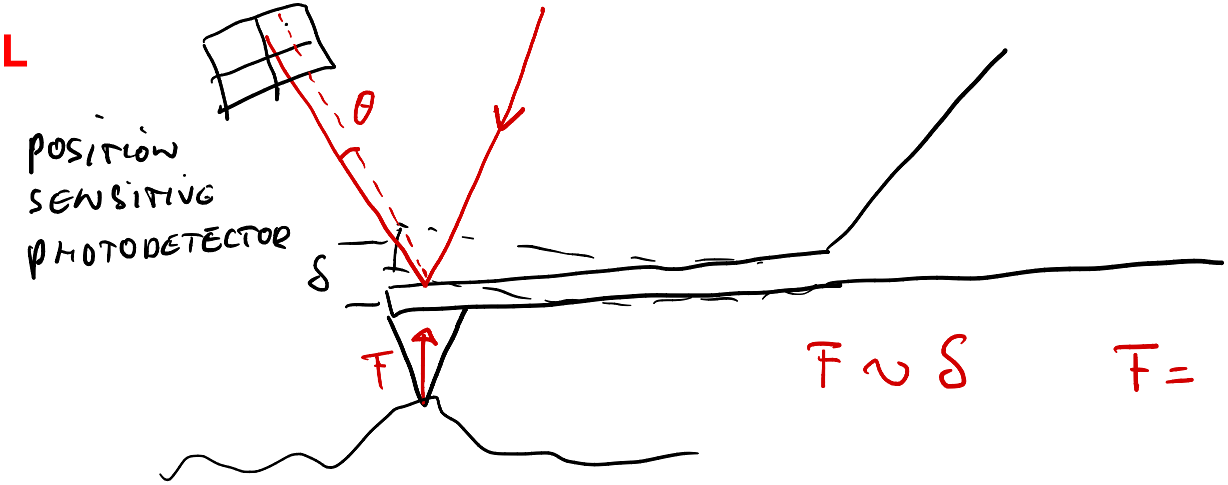
AFM-a versatile tool for nanoscale biology

- Single molecule resolution
- High resolution imaging in aqueous solution
- Imaging of living cells
- Single molecule mechanics
- Can be combined with optical microscopy





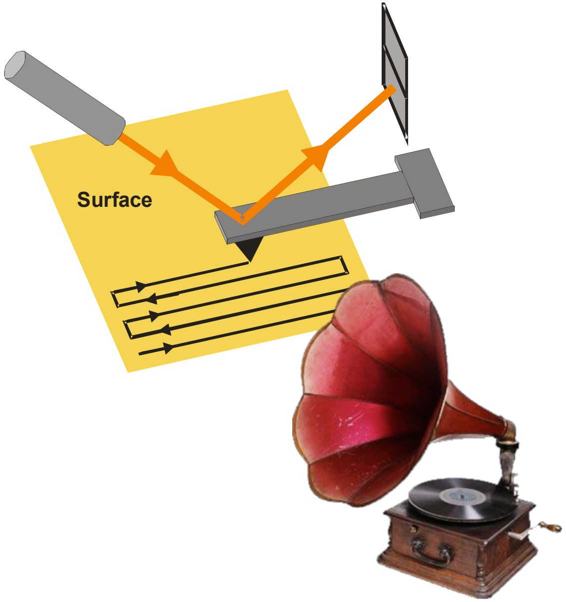


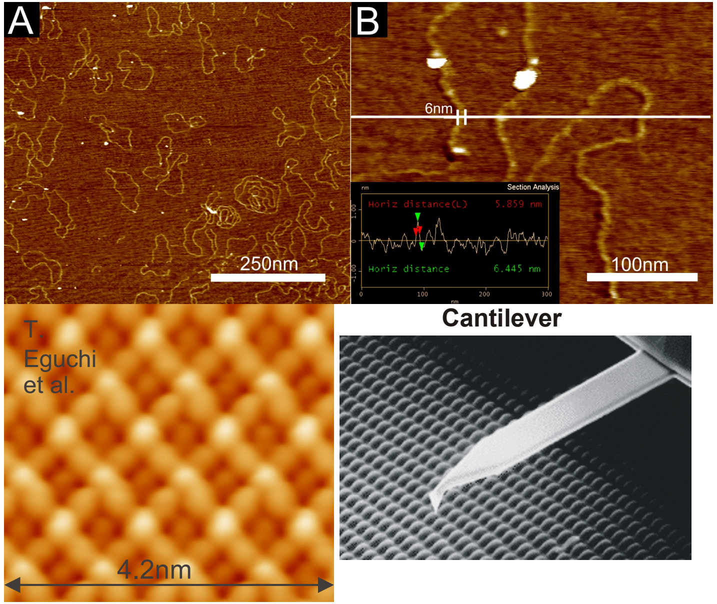


$F \sim \delta$ $\quad F = k \cdot \delta$

AFM: a Versatile Tool for Nanoscale Measurements

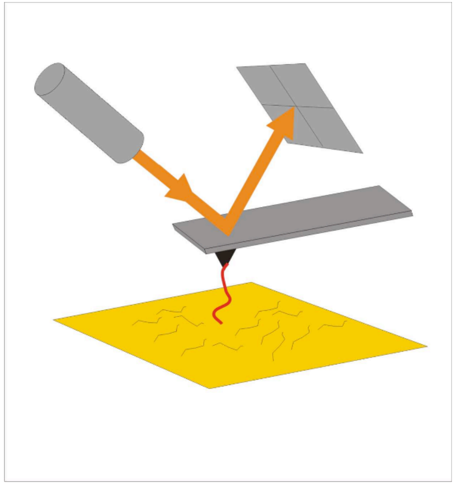
$1\,pN = 1 \cdot 10^{-12}\,N$

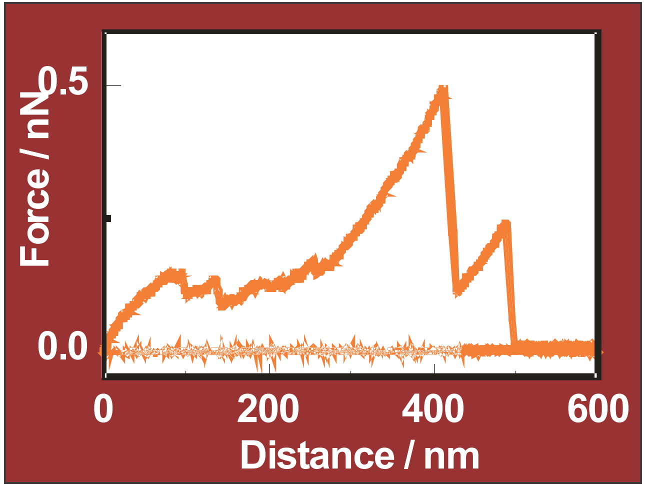




conductivity, surface potential, electrochemical potential,
ion currents, magnetism, NMR….and many more

Single Molecule Force Spectroscopy



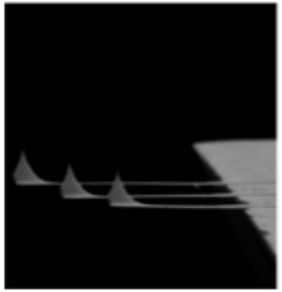


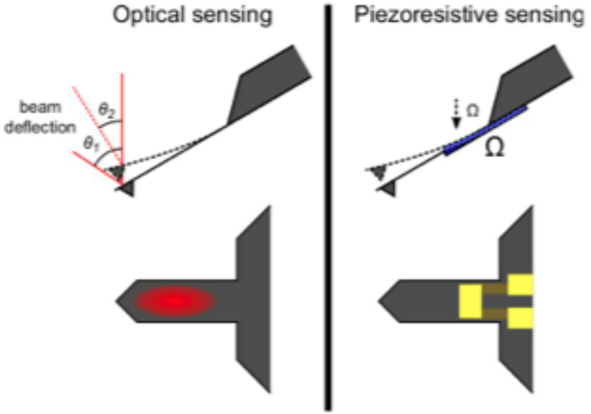
Force resolution: 10s of pN; limited by thermal motion of the cantilever

AFM cantilever beam

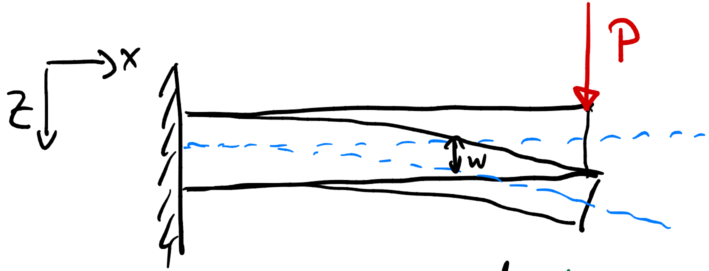
The gate to the nanoworld

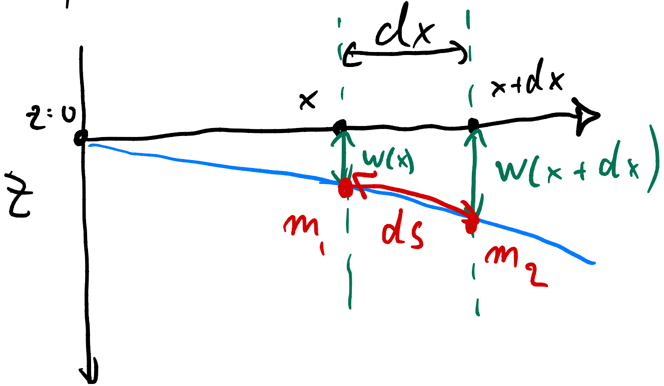
- In order to measure very fine features, the cantilever probe needs to be very sharp and sensitive
- The deflection of the cantilever has to be measured very precisely
- Two methods are often used:
  - Optical beam deflection
  - Piezoelectric strain sensing

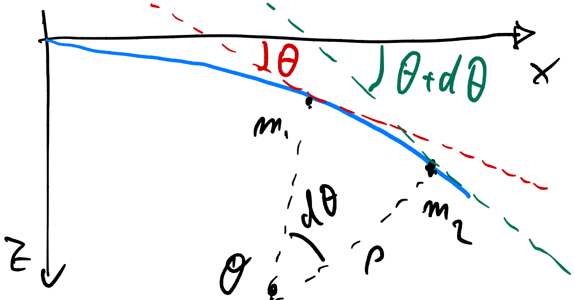




BEAM BENDING:







- THE LOAD P INDUCES A VERTICAL DISPLACEMENT OF THE CANTILEVER $w(x)$
- WE WANT TO FIND THE RELATIONSHIP BETWEEN P AND $w(x)$

DISPLACEMENT = DISTANCE FROM $z=0$

$$w(m_1) = w$$
$$w(m_2) = w + dw$$

ROTATION = ANGLE OF THE TANGENT WITH x-AXIS

$$\theta(m_1) = \theta$$
$$\theta(m_2) = \theta + d\theta$$

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- FOR SMALL ANGLES:

$$\rho \cdot d\theta \approx ds \quad \Rightarrow \quad \frac{d\theta}{ds} = \frac{1}{\rho} = \kappa \ \ldots \text{ CURVATURE}$$

- FOR SMALL DISPLACEMENTS: $ds \approx dx$

$$\boxed{\kappa = \frac{1}{\rho} = \frac{d\theta}{dx}}$$

- THE SLOPE θ AT ANY GIVEN POINT ON THE CURVE IS BY DEFINITION THE FIRST DERIVATIVE OF THE CURVE AT THAT POINT. SINCE WE MEASURED θ FROM THE x-AXIS DOWN WE GET $\theta = -\frac{dw}{dx}$ $\boxed{\frac{dw}{dx} = -\theta}$ EQN ①

$$\boxed{\kappa = \frac{1}{\rho} = -\frac{d^2w}{dx^2}}$$

- AS LONG AS BEAM BEHAVES LINEARLY, WE CAN APPLY Hooke's Law

$$\kappa = \frac{M}{EI}$$

$$\boxed{\frac{d^2w}{dx^2} = -\frac{M}{EI}} \quad \text{EQN (2)} \qquad M = -EI\frac{d^2w}{dx^2}$$

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- WE ALREADY KNOW:

$$\frac{dV}{dx} = -q \qquad \frac{dM}{dx} = V \Rightarrow \text{PUT INTO EQN } ②$$

$$V = \frac{dM}{dx} = \frac{d}{dx}\left(-EI\frac{d^2w}{dx}\right)$$

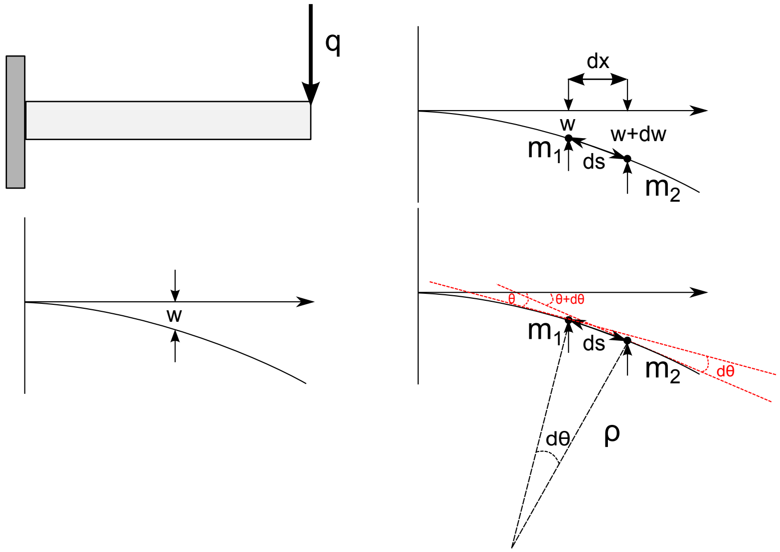
$$\boxed{V = -EI\frac{d^3w}{dx^3}} \quad \text{EQN } ③$$

FOR CONSTANT E & I

- IN THE SAME WAY WE GET:

$$\boxed{q = EI\frac{d^4w}{dx^4}} \quad \text{EQN } ④$$

Beam bending



- We bend the cantilever beam by applying a load at the end
- w(x) describes the amount of deflection of the point on the cantilever from the zero axis
- Two points are a distance ds apart from each-other on the bent beam
- From this we can get a relationship that describes the curvature of the beam

Beam bending - Governing equation

$$\frac{dw}{dx} = -\theta \quad (1)$$

$$\frac{d^2w}{dx^2} = -\frac{M(x)}{EI} \quad (2)$$

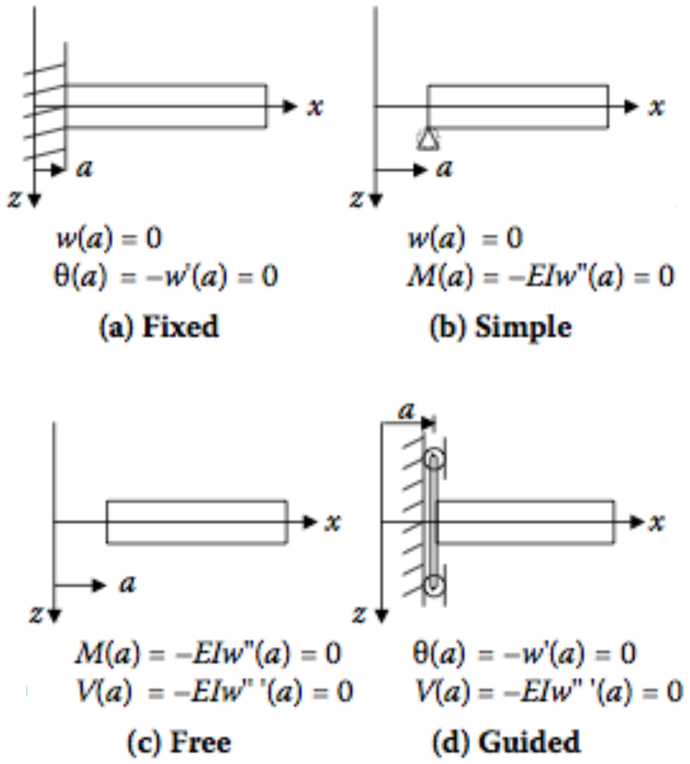
$$\frac{d^3w}{dx^3} = -\frac{V(x)}{EI} \quad (3)$$

$$\frac{d^4w}{dx^4} = \frac{q(x)}{EI} \quad (4)$$

We want to find a relationship between the beam deflection at a point x on the beam as a function of the load

We find 4 differential equations that relate loads to the deflection and the angle

Beam bending-Boundary condition



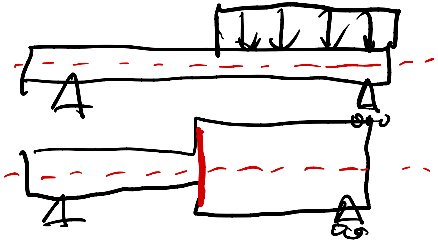
To solve for the beam bending equation through integration, we need boundary conditions

The type of support of the beam at its end determines the internal forces and moments at the ends, as well as its geometry

We have therefore two types of boundary conditions:

- Static boundary conditions: These come from static equilibrium and pertain to force related quantities (V,M)
- Kinematic boundary conditions: these define the deformational and geometric constraints for the angle and the bending

Beam bending - Abrupt changes



- When we have mathematical discontinuities due to an abrupt change in load or stiffness, we supplement our boundary conditions with the physical requirement that the neutral axis must be continuous!
- Deflection and tangent needs to be the same coming from both sides of the point of discontinuity:

$$\lim_{x \uparrow a} w(a) = \lim_{x \downarrow a} w(a)$$

$$\lim_{x \uparrow a} w'(a) = \lim_{x \downarrow a} w'(a)$$

Beam deflection

- Solving through integration

  - If we want to solve beam equation through integration, we need to integrate 4 times:

$$\int EIw''''(x)dx = \int q(x)dx$$

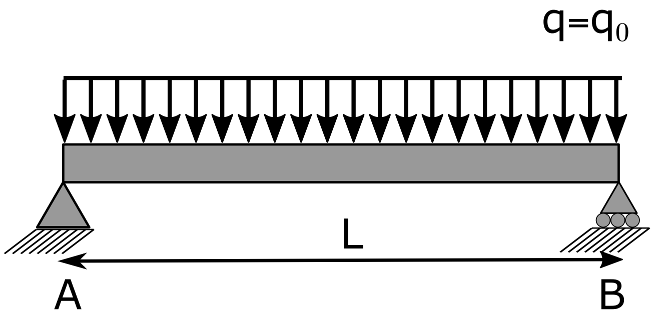
$$EIw(x) = \int\int\int\int q(x)dx + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4$$

  - We know already that:

$$V = -\int q(x)dx + C_1 \qquad M = -\int q(x)dx + C_1x + C_2$$

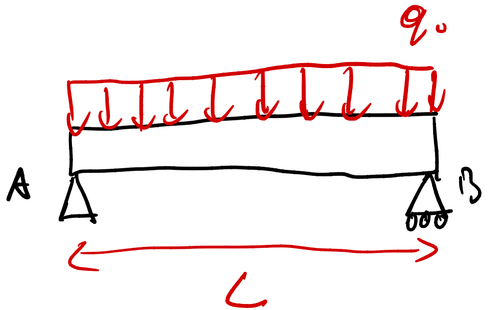
  - Therefore:
    - We get C1 and C2 from the boundary conditions of M(x) and V(x)
    - We get C3 from the boundary condition of the angle of deflection and C4 from the boundary condition of w

Example:
Simply supported beam in bending



A beam made of material *E* with a crossectional moment of area *I* is loaded with a load of constant laod density q_0. Calculate:

- The bending shape
- The maximum deflection
- The angles at the supports



■ GIVEN: GEOMETRY: – LENGTH L

– SIMPLE SUPPORTS

LOADS: – q = CONST

■ ASKED: 1 $w(x)$

2 δ_{MAX}

3 $\theta(A)$, $\theta(B)$

■ APPROACH: SINGULARITY FUNCTIONS & INTEGRATION

■ ANSWER:

$$q(x) = q_0$$

$$\frac{d^4 w}{d x^4} = \frac{q}{EI}$$

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$$\frac{d^4 w}{dx^4} = \frac{q}{EI} \qquad \int dx$$

$$\frac{d^3 w}{dx^3} = \frac{q}{EI} x + C_1 \qquad \int dx$$

$$\frac{d^2 w}{dx^2} = \frac{q}{2EI} x^2 + C_1 x + C_2 \qquad \int dx$$

$$\frac{dw}{dx} = \frac{q}{6EI} x^3 + \frac{C_1}{2} x^2 + C_2 x + C_3 \qquad \int dx$$

$$w(x) = \frac{q}{24EI} x^4 + \frac{C_1}{6} x^3 + \frac{C_2}{2} x^2 + C_3 x + C_4$$

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ENTER B.C.:

1 $w(0) = 0$ 3 $M(0) = 0 = w''(0)$

2 $w(L) = 0$ 4 $M(L) = 0 = w''(L)$

FROM 1: $C_4 = 0$

3: $C_2 = 0$

4: $\frac{q}{2EI} L^2 + C_1 L = 0 \Rightarrow C_1 = -\frac{q}{2EI} L$

2: $\frac{q}{24EI} L^4 - \frac{q}{12EI} L^4 + C_3 L = 0 \Rightarrow C_3 = \frac{qL^3}{24EI}$

$$w(x) = \frac{q}{24EI} x^4 - \frac{qL}{12EI} x^3 + \frac{qL^3}{24EI} x$$

$$\boxed{w(x) = \frac{q}{24EI}\left(x^4 - 2Lx^3 + L^3 x\right)}$$

2) δ_{MAX}: $w'(x) = 0$

$$w'(x) = \cancel{\frac{q}{24EI}}\left(4x^3 - 6Lx^2 + L^3\right) = 0$$

$$x = \frac{L}{2} \quad \vee \quad \cancel{x = -0{,}366L} \quad \vee \quad \cancel{x = 1{,}366L}$$

$$\delta_{MAX} = \frac{q}{24EI}\left(\frac{L^4}{16} - 2L\frac{L^3}{8} + L^3\frac{L}{2}\right)$$

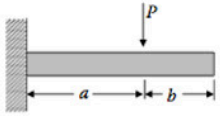
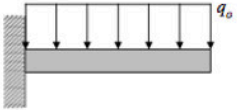
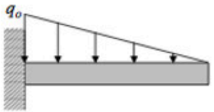
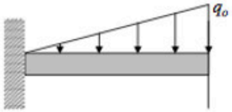
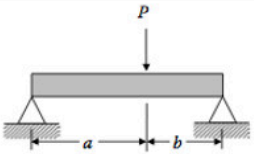
$$\boxed{\delta_{MAX} = \frac{5}{384}\frac{qL^4}{EI}}$$

3/ $\theta(x) = -w'(x) = -\frac{q}{24EI}\left(4x^3 - 6Lx^2 + L^3\right)$

$\theta(0) = -\frac{qL^3}{24EI}$

$\theta(L) = -\frac{q}{24EI}\left(4L^3 - 6L^3 + L^3\right) = \frac{qL^3}{24EI}$

Beam deflection – Solving through superposition

	$w(x)=\frac{PL^3}{6EI}\left[3\left(\frac{x}{L}\right)^2-\left(\frac{x}{L}\right)^3\right]$
	$w(x)=\frac{Pa^3}{6EI}\left[3\left(\frac{x}{a}\right)^2-\left(\frac{x}{a}\right)^3\right] \quad 0\le x\le a$ $w(x)=\frac{Pa^3}{6EI}\left[3\left(\frac{x}{a}\right)-1\right] \quad a\le x\le L$
	$w(x)=\frac{q_oL^4}{24EI}\left[6\left(\frac{x}{L}\right)^2-4\left(\frac{x}{L}\right)^3+\left(\frac{x}{L}\right)^4\right]$
	$w(x)=\frac{q_oL^4}{120EI}\left[10\left(\frac{x}{L}\right)^2-10\left(\frac{x}{L}\right)^3+5\left(\frac{x}{L}\right)^4-\left(\frac{x}{L}\right)^5\right]$
	$w(x)=\frac{q_oL^4}{120EI}\left[20\left(\frac{x}{L}\right)^2-10\left(\frac{x}{L}\right)^3+\left(\frac{x}{L}\right)^5\right]$
	$w(x)=\frac{PbL^2}{6EI}\left[\left(1-\left(\frac{b}{L}\right)^2\right)\frac{x}{L}-\left(\frac{x}{L}\right)^3\right] \quad 0\le x\le a$

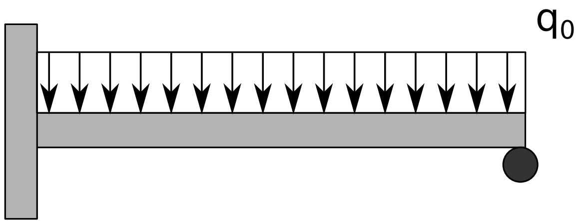
- As long as the beams behave linearly elastic, we are dealing with linear differential equations.
- For such a situation, we can separate a difficult load profile into simpler sub parts:

$$q(x) = q_1(x) + q_2(x) + \ldots$$

- We can then do the integrations over the individual q_i separately.
- To find the solution for the deflection due to the complex load profile, we can just sum up the deflections caused by the sub-loads q_i.

$$w(x) = \sum_i w_i(x)$$

- We can tabulate the deflection formulas due to standard loads.

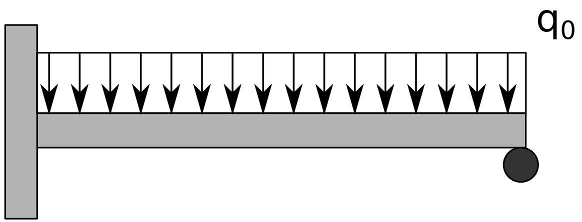


Statically indeterminate beams – Solving through integration

Often beams are supported more that absolutely required for static equilibrium.

A cantilever that is supported also on it's unmounted end is considered a "proper cantilever"

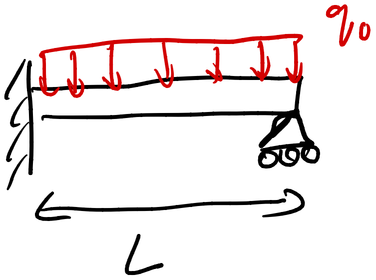
We treat over constrained beams in bending just like normal beams. The static indeterminacy is solved automatically through the use of the boundary conditions to calculate the integration constants.



Example: Statically indeterminate beams

- Solve the following statically indeterminate system through integration of the beam deflection differential equations. Calculate:
  - deflection
  - shear forces
  - bending moments
  - slopes

- Approach:
  - Set up load equation q(x)
  - Integrate the differential equations
  - Solve for the reaction forces using the boundary conditions



□ GIVEN: - GEOMETRY: • L

• BC @ $x=0$: FIXED

@ $x=L$: ROLLER

- LOAD: $q_0 = \text{CONST}$

□ ASKED: $w(x), V(x), M(x), \theta(x)$

□ APPROACH: INTEGRATION OF q

ANSWER:

$$q(x) = q_0 \langle x \rangle^0$$

$$\frac{d^4 w}{dx^4} = \frac{q}{EI}$$

$$\frac{d^3 w}{dx^3} = \frac{q}{EI} x + C_1$$

$$\frac{d^2 w}{dx^2} = \frac{q}{2EI} x^2 + C_1 x + C_2$$

$$\frac{dw}{dx} = \frac{q}{6EI} x^3 + \frac{C_1}{2} x^2 + C_2 x + C_3$$

$$w(x) = \frac{q}{24EI} x^4 + \frac{C_1}{6} x^3 + \frac{C_2}{2} x^2 + C_3 x + C_4$$

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B.C.:

① $w(0) = 0 \Rightarrow C_4 = 0$

② $\theta(0) = 0 \Rightarrow w'(0) = 0 \Rightarrow C_3 = 0$

③ $w(L) = 0$

④ $M(L) = 0 \Rightarrow w''(L) = 0$

From ④: $\dfrac{d^2 w}{d x^2} = \dfrac{q}{2EI} x^2 + C_1 x + C_2 = 0 \quad @ \; x = L$

$$\frac{qL^2}{2EI} + C_1 L + C_2 = 0$$

$$C_2 = -\left(\frac{q}{2EI} L^2 + C_1 L\right)$$

From B.C. ③ : $w(L) = 0$

$$w(x) = \frac{q}{24EI}x^4 + \frac{c_1}{6}x^3 + \frac{c_2}{2}x^2 + \cancel{C_3 x + C_4}$$

$$w(L) = \frac{q}{24EI}L^4 + \frac{c_1}{6}L^3 - \frac{1}{2}\left(\frac{q}{2EI}L^2 + c_1 L\right)L^2 = 0$$

$$\frac{q}{24EI}L^4 + \frac{c_1}{6}L^3 - \frac{6q}{24EI}L^4 - \frac{3c_1}{6}L^3 = 0$$

$$-\frac{5q}{24EI}L^{\cancel{4}} - \frac{2}{6}c_1 \cancel{L^3} = 0$$

$$C_1 = -3 \cdot \frac{5}{24EI}Lq = -\frac{5}{8EI}qL$$

$$C_2 = -\left(\frac{q}{2EI}L^2 + C_1 L\right) = -\frac{q}{2EI}L^2 + \frac{5}{8EI}qL^2 = \frac{1}{8EI}qL^2$$

$$w(x) = \frac{q}{24EI} x^4 + \frac{c_1}{6} x^3 + \frac{c_2}{2} x^2 + C_3 x + C_4$$

$$C_1 = -3 \cdot \frac{5}{24EI} L q = -\frac{5}{8EI} q L$$

$$C_2 = -\left(\frac{q}{2EI} L^2 + C_1 L\right) = -\frac{q}{2EI} L^2 + \frac{5}{8EI} q L^2 = \frac{1}{8EI} q L^2$$

$$\boxed{w(x) = \frac{q_0}{24EI} x^4 - \frac{5}{48EI} q_0 L x^3 + \frac{1}{16EI} \cdot q_0 L^2 x^2}$$

DEFL: $w(x) = \frac{q_0}{48EI} \left\{ 2x^4 - 5Lx^3 + 3L^2x^2 \right\}$

ANGLE: $\theta(x) = -\frac{q_0}{48EI} \left\{ 8x^3 - 15Lx^2 + 6L^2x \right\}$

MOMENT: $M(x) = -\frac{q_0}{48} \left\{ 24x^2 - 30Lx + 6L^2 \right\}$

SHEAR: $V(x) = -\frac{q_0}{48} \left\{ 48x - 30L \right\}$

LOAD: $\frac{dV}{dx} = -q \Rightarrow q(x) = +\frac{q_0}{48} \cdot 48 = q_0$ ✓

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